THE EXPECTED NUMBER OF CHILDREN

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Abstract. Compute the expectations of a family of random variables using generating functions.

1. Problem

A couple is determined to have at least one son and at least one daughter. What is the expected total number of children they will have? What if they want at least two sons and at leat one daughter?

2. Solution

Let the probability of the child being male be p and the probability of the child being female be $q = 1 - p$. Let the random variable $\xi_{a,b}$ be the number of children born until there are at least a boys and at least b girls. Let $P(X)$ be the probability of the event X. Then

$$\begin{aligned}
P(\xi_{a,0} = n) &= p\binom{n-1}{a-1}p^{a-1}q^{n-a} \\
&= \binom{n-1}{a-1}p^{a}q^{n-a} \\
P(\xi_{0,b} = n) &= \binom{n-1}{b-1}p^{n-b}q^{b} \\
P(\xi_{a,b} = a+b) &= \binom{a+b}{a}p^{a}q^{b} \\
P(\xi_{a,b} = n) &= qP(\xi_{a,b-1} = n-1) + pP(\xi_{a-1,b} = n-1)
\end{aligned}$$

Define the generating function

$$f_{a,b}(\lambda) = \sum_{n \geq a+b} P(\xi_{a,b} = n)\lambda^n$$

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Then

$$f'_{a,b}(1) = M(\xi_{a,b})$$

thus computing the expected value of $\xi_{a,b}$ is reduced to computing the derivative of $f_{a,b}$.

From the formulas above one can easily see

$$\begin{aligned}
f_{a,b}(\lambda) &= \lambda q f_{a,b-1}(\lambda) + \lambda p f_{a-1,b}(\lambda) \\
f_{a,0}(\lambda) &= \sum_{n \geq a} \lambda^n \binom{n-1}{a-1} p^a q^{n-a} \\
&= \lambda^a p^a \sum_{k \geq 0} \binom{a+k-1}{a-1} (\lambda q)^k \\
&= \left(\frac{\lambda p}{1 - \lambda q} \right)^a \\
f_{0,b}(\lambda) &= \left(\frac{\lambda q}{1 - \lambda p} \right)^b \\
f_{1,1}(\lambda) &= \lambda^2 pq \left(\frac{1}{1-\lambda p} + \frac{1}{1-\lambda q} \right) \\
f_{2,1}(\lambda) &= \lambda^3 p^2 q \left(\frac{1}{1-\lambda p} + \frac{1}{1-\lambda q} + \frac{1}{(1-\lambda q)^2} \right) \\
f_{2,2}(\lambda) &= \lambda^4 p^2 q^2 \left(\frac{2}{1-\lambda p} + \frac{2}{1-\lambda q} + \frac{1}{(1-\lambda p)^2} \frac{1}{(1-\lambda q)^2} \right)
\end{aligned}$$

Taking the derivatives of the last two formulas with respect to λ, we get the answers to the questions of the section 1:

$$\begin{aligned}
f'_{1,1}(\lambda) &= 2pq\lambda \left(\frac{1}{1-\lambda p} + \frac{1}{1-\lambda q} \right) + \\
&\quad pq\lambda^2 \left(\frac{p}{(1-\lambda p)^2} + \frac{q}{(1-\lambda q)^2} \right) \\
f'_{1,1}(1) &= 2 + \frac{p^3+q^3}{pq} \\
f'_{1,1}(1)|_{p=q=\frac{1}{2}} &= 3
\end{aligned}$$

and

$$\begin{aligned}
f'_{2,1}(\lambda) &= 3\lambda^2 p^2 q \left(\frac{1}{1-\lambda p} + \frac{1}{1-\lambda q} + \frac{1}{(1-\lambda q)^2} \right) + \\
&\quad \lambda^3 p^2 q \left(\frac{p}{(1-\lambda p)^2} + \frac{q}{(1-\lambda q)^2} + \frac{2q}{(1-\lambda q)^3} \right) \\
f'_{2,1}(1) &= p^2 q \left(\frac{3}{p} + \frac{3}{q} + \frac{3}{p^2} + \frac{p}{q^2} + \frac{q}{p^2} + \frac{2q}{p^3} \right) \\
f'_{2,1}(1)|_{p=q=\frac{1}{2}} &= 4\frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
f'_{2,2}(\lambda) &= 4\lambda^3 p^2 q^2 \left(\frac{2}{1-\lambda p} + \frac{2}{1-\lambda q} + \frac{1}{(1-\lambda p)^2} + \frac{1}{(1-\lambda q)^2} \right) + \\
&\quad 2\lambda^4 p^2 q^2 \left(\frac{p}{(1-\lambda p)^2} + \frac{q}{(1-\lambda q)^2} + \frac{p}{(1-\lambda p)^3} + \frac{q}{(1-\lambda q)^3} \right) \\
f'_{2,2}(1) &= 4p^2 q^2 \left(\frac{2}{q} + \frac{2}{p} + \frac{1}{q^2} + \frac{1}{p^2} \right) + \\
&\quad 2p^2 q^2 \left(\frac{p}{q^2} + \frac{q}{p^2} + \frac{p}{q^3} + \frac{q}{p^3} \right) \\
f'_{2,2}(1)|_{p=q=\frac{1}{2}} &= 5\frac{1}{2}
\end{aligned}$$

3. General Case

Define

$$F(x,y,\lambda) = \sum_{a,b\geq 0} f_{a,b}(\lambda) x^a y^b$$

Then

$$\begin{aligned}
F - \lambda p x F - \lambda q y F &= \\
1 + \sum_{a\geq 1} x^a (f_{a,0} - \lambda p f_{a-1,0}) + \sum_{b\geq 1} y^b (f_{0,b} - \lambda q f_{0,b-1}) &= \\
1 + \sum_{a\geq 1} \left(\frac{\lambda p x}{1-\lambda q} \right)^a \lambda q + \sum_{b\geq 1} \left(\frac{\lambda q y}{1-\lambda p} \right)^b \lambda p &= \\
1 + \lambda q \left(\frac{1}{1-\frac{\lambda p x}{1-\lambda q}} - 1 \right) + \lambda p \left(\frac{1}{1-\frac{\lambda q y}{1-\lambda p}} - 1 \right)
\end{aligned}$$

Therefore

$$F = \frac{1 + \lambda^2 pq \left(\frac{x}{1-\lambda q-\lambda px} + \frac{y}{1-\lambda p-\lambda qy} \right)}{1 - \lambda px - \lambda qy}$$

Hmmm - this is not so easy after all...

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